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A DYADIC AGE-REPLACEMENT POLICY FOR A PERIODICALLY
INSPECTED EQUIPMENT ITEM SUBJECT
TO RANDOM DETERIORATION

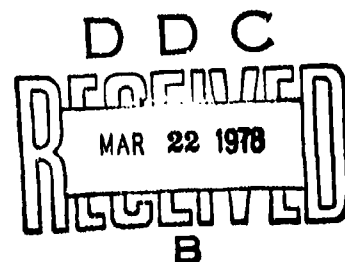Research Report No. 77-6

by

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July 1977

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age and the chronological age and for the distribution of the total number N_t of replacements are derived. The derivation of the distribution function of N_t relies on the solution to a system of linear Diophantine equations. Finally, using as criterion the minimization of the total steady-state expected cost per period, consisting of a fixed replacement cost and a linear cost of operation, optimal values of S and N are computed for a few numerical examples.

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Abstract

A periodic review replacement system is considered. The amount of deterioration over successive periods form a sequence of i.i.d. random variables. A replacement policy of the dyadic type is in effect whereby the used equipment item is discarded and immediately replaced by a new identical equipment item if at the end of a period the old equipment has service aged by an amount in excess of S or has been in operation for exactly N periods whichever comes first. Expressions for the joint distribution of the service age and the chronological age and for the distribution of the total number of replacements N_t are derived. The derivation of the distribution function of N_t relies on the solution to a system of linear Diophantine equations. Finally, using as criterion the minimization of the total steady-state expected cost per period, consisting of a fixed replacement cost and a linear cost of operation, optimal values of S and N are computed for a few numerical examples.

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1. Introduction

We consider a periodic review replacement system of a piece of equipment item which ages while in operation. The amount of deterioration or service aging $\{D_i\}$ over successive equally spaced periods ($i = 1, 2, \ldots$) form a sequence of independent and identically distributed positive random variables with know distribution function $\Phi(\xi)$ and p.d.f. $\phi(\xi), 0 < \xi < \infty$. We assume that a replacement policy of the dyadic type is in effect whereby at the end of each period the total amount of deterioration of the equipment item since acquisition (service aging) is measured as well as the total number of periods elapsed since such acquisition (chronological aging); the used equipment item is discarded and immediately replaced by a new identical equipment item if at the end of a period the old equipment has service aged by an amount in excess of S or has been in operation for exactly N periods whichever comes first. We shall denote this policy as an (S, N) policy.

Age replacement problems have been studied by several authors in the past ([1], [2], [4], [9], [11]), although these chiefly center around deterioration and/or breakdown in continuous time. In discrete time maintenance models, the approach has been mostly based on Markov decision theory [3], [6], [7]. In this respect, the model considered in this paper departs from practice in that it studies a two decision variable (dyadic) periodic review age replacement problem using renewal theory.

Consider for example an equipment item which is operating intermittently characterized by the fact that over equal time intervals the total service provided varies depending upon the user's request for service. Two actors determine the deterioration of the equipment item, namely, its actual usage and the time

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elapsed since acquisition. The equipment may be a copier in a duplicating office. Its usage over successive intervals of months is measured by the total number of reproductions accumulated during the month. A maintenance policy is then dictated simultaneously by the total number S of reproductions since last maintenance (approximated by a continuous variable) as well as the time N elapsed since such maintenance. Another example is provided by a road vehicle in which the usage is measured by the total number of miles registered during successive unit time intervals. The age of the road vehicle is both a function of the total number of miles registered as well as the time elapsed since acquisition. The road vehicle is replaced whenever its usage exceeds a given level S or whenever it has been in operation for a certain length of time N whichever comes first.

We assume that at time origin, $t = 0$, the equipment item has just been replaced and has service and chronological ages equal to zero. Following a decision at beginning of period t, $t = 1, 2, \ldots$, let

N_t = total number of replacements in the time interval $(0, t]$

Y_t = service age of equipment item, $0 \leq Y_t < S$

0_t = chronological age of equipment item,

$0_t = 0, 1, \ldots, N - 1$

In what follows we shall consider the two stochastic processes $\{(Y_t, 0_t), t = 1, 2, \ldots\}$ and $\{N_t, t = 1, 2, \ldots\}$ and derive in particular expressions for the joint distribution function of Y_t and 0_t and for the distribution function of N_t. Since the case when $N = 1$ is trivial, we restrict ourselves in the sequel to the case when $N \geq 2$. In deriving the distribution function of N_t, the analysis will rely on the solution to a system of linear Diophantine equations. Finally, using as criterion the minimization of the total steady state expected cost per period, consisting of a fixed replacement cost and a linear cost of operation, optimal values of S and N are computed for cases when $\{D_i\}$ have gamma distributions.

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2. Distribution of the Waiting Time Till the kth Replacement.

Let $\{T_j\}$, $j = 1, 2, \ldots$ represent the sequence of interarrival times between replacements. $\{T_j\}$ is an ordinary renewal process over discrete times [8]. The distribution function of T_j is given by

$$P\{T_j \leq 0\} = 0$$

$$P\{T_j \leq n\} = P\{D_1 + D_2 + \cdots + D_n > S\}$$

$$= 1 - \phi^{(n)}(S) \quad , \qquad n = 1, 2, \ldots, N - 1 \tag{1}$$

$$P\{T_j \leq N\} = 1$$

where we define

$$\phi^{(n)}(x) = \int_0^x \phi^{(n-1)}(x - u) \, d\phi(u) \quad , \; n = 1, 2, \ldots; (\phi^{(0)}(\cdot) = 1) \tag{2}$$

Let $f_{T_j}(n)$ be the probability mass function of the interarrival time T_j. It immediately follows from (1) that

$$f_{T_j}(n) = \begin{cases} \phi^{(n-1)}(S) - \phi^{(n)}(S) & n = 1, 2, \ldots, N - 1 \\ \phi^{(n-1)}(S) & n = N \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

and

$$E[T_j] = \sum_{-\infty}^{\infty} n f_{T_j}(n)$$

$$= 1 + \sum_{r=1}^{r=N-1} \phi^{(r)}(S) \tag{4}$$

Also, the probability generating function of T_j, $G_{T_j}(u)$, may be evaluated:

$$G_{T_j}(u) = \sum_{-\infty}^{\infty} u^n f_{T_j}(n) \qquad\qquad |u| \leq 1 \tag{5}$$

$$= u - (1 - u) \sum_{n=1}^{n=N-1} u^n \phi^{(n)}(S)$$

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Let W_k be the waiting time till the kth replacement $k = 1, 2, \ldots$. The probability mass function of W_k is clearly

$$f_{W_k}(t) = f_{T_j}^{*(k)}(t) \tag{6}$$

where * denotes the usual convolution operation. Thus, the probability generating function of W_k, $G_{W_k}(u)$ is

$$G_{W_k}(u) = [u - (1 - u) \sum_{n=1}^{N-1} u^n \phi^{(n)}(S)]^k, \quad k = 1, 2, \ldots \tag{7}$$

from which the distribution function of W_k can be derived.

3. Joint Distribution Function of Y_t and Θ_t

Let for $t = 1, 2, \ldots$

M_t = probability of a replacement being made at beginning of time t.

$\Psi_t(y, \theta)dy$ = probability that at beginning of time t, the service age of the equipment item lies between y and $y + dy$ and its chronological age is exactly θ, $0 < y < S$ and $\theta = 1, 2, \ldots, N - 1$.

Thus,

$$M_t = P\{Y_t = 0; \; \Theta_t = 0\} \tag{8}$$

$$= \sum_{k=1}^{\infty} P\{W_k = t\} = \sum_{k=1}^{\infty} f_{T_j}^{*(k)}(t), \quad t = 1, 2, \ldots$$

Also, for $0 < y < S$ and $\theta = 1, 2, \ldots, N - 1$

$$\Psi_t(y, \theta)dy = P\{y < Y_t \leq y + dy; \; \Theta_t = \theta\} \tag{9}$$

$$= \sum_{k=1}^{\infty} P\{y < Y_t \leq y + dy \,|\, W_k = t - \theta\} . \; P\{W_k = t - \theta\}$$

$$= \sum_{k=1}^{\infty} d \, \phi^{(\theta)}(y) \, f_{T_j}^{*(k)}(t-\theta) \quad , \; t = 1, 2, \ldots$$

Some well known results in renewal theory [8] can be obtained from relations (3) to (9) by considering the special case when N equals to ∞.

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4. Limiting Distribution of $\Psi_t(y, \theta)$

It is easily verified that

$$M = \lim_{t \to \infty} M_t = \frac{1}{E[T_j]} = \frac{1}{1 + \sum_{r=1}^{N-1} \phi^{(r)}(S)} \tag{10}$$

To determine $\Psi(y, \theta)dy = \lim_{t \to \infty} \Psi_t(y, \theta)dy$, where $0 < y < S$ and $\theta = 1, 2, \ldots, N - 1$, define the generating function

$$G(u) = \sum_{t=1}^{\infty} u^t \Psi_t(y, \theta)dy \qquad |u| \leq 1$$

Using (9) we obtain

$$G(u) = \sum_{t=1}^{\infty} u^t \sum_{k=1}^{\infty} d\, \phi^{(\theta)}(y)\, f_{T_j}^{*(k)}(t - \theta)$$

$$= u^\theta [d\, \phi^{(\theta)}(y)]\, [\frac{1}{1 - G_{T_j}(u)} - 1]$$

Then,

$$\Psi(y, \theta)dy = \lim_{u \to 1} (1 - u)u^\theta [d\, \phi^{(\theta)}(y)]\, [\frac{1}{1 - G_{T_j}(u)} - 1]$$

$$= \frac{d\, \phi^{(\theta)}(y)}{E[T_j]}$$

$$= \frac{d\, \phi^{(\theta)}(y)}{1 + \sum_{r=1}^{N-1} \phi^{(r)}(S)}, \quad 0 < y < S, \ \theta = 1, 2, \ldots, N - 1 \tag{11}$$

Limiting distribution under an (S, ∞) policy follows immediately from (10) and (11). It is also possible to determine from (10) and (11) the marginal distribution of service age at beginning of period and end of period.

5. The Distribution of N_t, $t = 1, 2, \ldots$

Let $G_{N_t}(z, t)$ be the probability generating function of N_t, i.e.,

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$$G_{N_t}(z, t) = \sum_{r=0}^{\infty} z^r \, P\{N_t = r\} \qquad , \quad |z| \le 1 \tag{12}$$

Also, let $G(z, w)$ be the generating function of $G_{N_t}(z, t)$ with respect to t , i.e.,

$$G(z, w) = \sum_{t=1}^{\infty} w^t \, G_{N_t}(z, t) \qquad , \quad |w| < 1 \tag{13}$$

Then, [8]

$$G(z, w) = \frac{1 - G_{T_j}(w)}{(1 - w)[1 - z \, G_{T_j}(w)]} \tag{14}$$

and, using (5), we obtain

$$G(z, w) = [1 + \sum_{n=1}^{N-1} w^n \, \phi^{(n)}(S)] \, [\sum_{m=0}^{\infty} \{z \, G_{T_j}(w)\}^m] \tag{15}$$

5.1 Inversion of $G(z, w)$

In order to determine $P\{N_t = r\}$, $r = 0, 1, 2, \ldots$, we first invert $G(z, w)$ with respect to w and the resultant expression with respect to z.

For notational convenience, let

$$b_n = \phi^{(n)}(S) \quad n = 1, 2, \ldots \tag{16}$$
$$b_0 = 1$$

Then (5) becomes

$$G_{T_j}(u) = [\sum_{i=1}^{N-1} u^i (b_{i-1} - b_i)] + u^N b_{N-1} \tag{17}$$

and (15) becomes

$$G(z, w) = [b_0 + \sum_{n=1}^{N-1} w^n b_n] \, [\sum_{m=0}^{\infty} z^m \{G_{T_j}(w)\}^m] \tag{18}$$

To derive the coefficient of w^t in $G(z, w)$, $t = 1, 2, \ldots$, it is necessary to develop a systematic procedure for determining the coefficient of w^τ in the multinomial expansion of an expression of the form $(a_1 w + a_2 w^2 + \cdots + a_N w^N)^\upsilon$.

where the a_i, $i = 1, \ldots, N$, are known constants, υ is any positive integer,

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and $\tau = 1, 2, \ldots, \upsilon N$.

From the Multinomial Theorem, the coefficient of w^τ is given by

$$\sum \frac{\upsilon!}{n_1! n_2! \cdots n_N!} a_1^{n_1} a_2^{n_2} \cdots a_N^{n_N} \tag{19}$$

where the summation is taken over all non-negative integers, n_i, $i = 1, 2, \ldots, N$ such that

$$n_1 + n_2 + \cdots + n_N = \upsilon \tag{20}$$

$$n_1 + 2n_2 + \cdots + Nn_N = \tau \tag{21}$$

If no set of n_i, $i = 1, 2, \ldots, N$ exists satisfying (20), (21) then the coefficient is equal to zero.

5.2 On the Solution of a System of Two Linear Diophantine Equations

(20), (21) represent a system of two linear Diophantine equations in N non-negative integral unknowns. It has been shown, [5], that any system of m linear Diophantine equations can be transformed into a single equation and a set of inequalities. For our special structure, the following interesting theorem is obtained by applying the so-called 'Rule of Virgins' (method of eliminating one unknown at a time) [10].

Theorem: The system of two linear Diophantine equations in N (> 1) non-negative integral unknowns

$$n_1 + n_2 + \cdots + n_N = \upsilon \tag{20}$$

$$n_1 + 2n_2 + \cdots + Nn_N = \tau \tag{21}$$

where υ and τ are positive integers, has a solution if and only if $\upsilon \le \tau \le \upsilon N$.

If it has a solution, then all possible solutions of (20), (21) are generated by all possible solutions of

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$$c_1 + c_2 + \cdots + c_{N-1} = \upsilon - \tau + (N - 1) [\tau/N] \tag{22}$$

and

$$[\tau/N] - \upsilon \leq c_{N-1} \leq c_{N-2} \leq \cdots \leq c_2 \leq c_1 \leq [\tau/N] \tag{23}$$

where the c_i, $i = 1, 2, \ldots, N - 1$ are integers (unrestricted in sign) and $[\tau/N]$ is the greatest integer $\leq \tau/N$; and the n_i, $i = 1, 2, \ldots, N$, are given by

$$n_1 = \upsilon - [\tau/N] + c_{N-1} \tag{24}$$

$$n_{N-j} = c_j - c_{j+1} \qquad , j = 1, 2, \ldots, N - 2$$

$$n_N = [\tau/N] - c_1$$

Further, if the original system of equations has no solution, then there is no solution to (22), (23), so that the two systems are equivalent.

<u>Proof</u>: It is easily verified that if $\tau < \upsilon$ or if $\tau > \upsilon N$, then there is no solution to the system of two linear Diophantine equations. Also, if $\upsilon \leq \tau \leq \upsilon N$, it is clear that the coefficient of w^τ in $(a_1 w + a_2 w^2 + \cdots + a_N w^N)^\upsilon$ is given by (19) (this coefficient being not necessarily non-zero); so that there is at least one solution to (20), (21).

To prove the second part of the theorem, assume that there exist n_i, $i = 1, 2, \ldots, N$ satisfying (20), (21). Then from (21) we have,

$$n_N = \tau/N - (\sum_{i=1}^{N-1} in_i)/N$$

$$= [\tau/N] - (\sum_{i=1}^{N-1} in_i + N[\tau/N] - \tau)/N$$

Now let

$$c_1 = (\sum_{i=1}^{N-1} in_i + N[\tau/N] - \tau)/N \tag{25}$$

Clearly, c_1 is an integer, and n_N being non-negative, $c_1 \leq [\tau/N]$. Also,

$$n_N = [\tau/N] - c_1 \tag{26}$$

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Again, from (25), we have,

$$n_{N-1} = c_1 - (\sum_{i=1}^{N-2} in_i - c_1 + N[\tau/N] - \tau)/(N-1)$$

Letting $c_2 = (\sum_{i=1}^{N-2} in_i - c_1 + N[\tau/N] - \tau)/(N-1)$, we get

$$n_{N-1} = c_1 - c_2$$

Clearly, c_2 is an integer, $c_2 \leq c_1$

Proceeding in this fashion, we obtain

$$n_{(N-j)} = c_j - c_{j+1} \quad j = 1, 2, \ldots, N-2 \tag{27}$$

where $c_j = (\sum_{i=1}^{N-j} in_i - \sum_{i=1}^{j-1} c_i + N[\tau/N] - \tau)/(N-j+1) \quad j = 2, 3, \ldots, N-1$ (28)

and $c_j \leq c_{j-1} \quad j = 2, 3, \ldots, N-1$

Letting $j = N-1$ in (28), we obtain

$$n_1 = \sum_{i=1}^{N-1} c_i + c_{N-1} - N[\tau/N] + \tau \tag{29}$$

From (20), (26), (27) and (29), we have,

$$\sum_{i=1}^{N-1} c_i = \upsilon - \tau + (N-1)[\tau/N]$$

so that, from (29),

$$n_1 = \upsilon - [\tau/N] + c_{N-1}$$

and since $n_1 \geq 0$, $[\tau/N] - \upsilon \leq c_{N-1}$

Thus, starting from (20), (21) we have obtained (22), (23) and (24). Finally, it can be easily shown that any solution to (22), (23) yields a solution to (20), (21) given by (24), so that the two systems are equivalent.

Q.E.D.

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To illustrate the use of the theorem we solve

$$n_1 + n_2 + n_3 = 5$$

$$n_1 + 2n_2 + 3n_3 = 8$$

Here, $N = 3$, $\upsilon = 5$, $\tau = 8$ and $[\tau/N] = [8/3] = 2$

From (22), (23),

$$c_1 + c_2 = 5 - 2 - 2 = 1$$

subject to $-3 \leq c_2 \leq c_1 \leq 2$

which yields as solutions, $\{c_1 = 2, c_2 = -1\}$ and $\{c_1 = 1, c_2 = 0\}$.

The corresponding solutions to the original system are from (24):

$$\{n_1 = 2, n_2 = 3, n_3 = 0\} \text{ and } \{n_1 = 3, n_2 = 1, n_3 = 1\}$$

It should be noted here that the computational effort required to solve
the transformed system is no greater than that for the original system. In
fact, in many instances, especially when the number of solutions is not large,
the transformed system is easier to solve, being more amenable to an enumerative
scheme. Thus, although the theorem presented is interesting in itself, it also
possesses some utility.

5.3 An Expression for $P\{N_t = r\}$ $t = 1, 2, \ldots$; $r = 0, 1, 2, \ldots$

Let $I_{\upsilon,\tau}$ be the number of solutions to the system of linear Diophantine
equations (20), (21). Let $\{n_{ij}, i = 1, 2, \ldots, N\}$ be the jth solution,
$j = 1, 2, \ldots, I_{\upsilon,\tau}$, if $I_{\upsilon,\tau} > 0$. Define

$$A_{\upsilon,\tau,j} = \frac{a_1^{n_{1j}} a_2^{n_{2j}} \ldots a_n^{n_{Nj}}}{n_{1j}! \, n_{2j}! \, \ldots \, n_{Nj}!} , \quad j = 1, 2, \ldots, I_{\upsilon,\tau}; \ I_{\upsilon,\tau} > 0 \qquad (30)$$

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Then, from (19), the coefficient of w^τ in $(a_1 w + a_2 w^2 + \cdots a_N w^N)^\upsilon$ is given by

$$\upsilon! \sum_{j=1}^{I_{\upsilon,\tau}} A_{\upsilon,\tau,j} \qquad (31)$$

If $I_{\upsilon,\tau} = 0$, the above expression is defined to be zero.

Let

$$a_i = b_{i-1} - b_i, \quad i = 1, 2, \ldots, N-1 \qquad (32)$$

$$a_N = b_{N-1}$$

Then (17) becomes

$$G_{T_j}(u) = a_1 u + a_2 u^2 + \cdots + a_N u^N \qquad (33)$$

and hence it follows from (31) that the coefficient of w^t, $t = 1, 2, \ldots$ in $[G_{T_j}(w)]^k$, $k = 1, 2, \ldots$, is given by

$$k! \sum_{j=1}^{I_{k,t}} A_{k,t,j} \qquad (34)$$

From (18), it is easily seen that when $1 < N \leq t$, the coefficient of w^t in $G(z, w)$ is given by

$$\sum_{q=t-N+1}^{t} b_{t-q} \sum_{k=1}^{q} z^k \{\text{Coefficient of } w^q \text{ in } [G_{T_j}(w)]^k\} \qquad (35)$$

and for all values of N greater than t, the coefficient of w^t is given by

$$b_t + \sum_{q=1}^{t} b_{t-q} \sum_{k=1}^{q} z^k \{\text{Coefficient of } w^q \text{ in } [G_{T_j}(w)]^k\} \qquad (36)$$

Thus, from (34), (35), (36), we get

$$\sum_{r=0}^{\infty} z^r P\{N_t = r\} = \sum_{q=t-N+1}^{t} b_{t-q} \sum_{k=1}^{q} k! \, z^k \sum_{j=1}^{I_{k,q}} A_{k,q,j}, \quad 1 < N \leq t \qquad (37)$$

and

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$$\sum_{r=0}^{\infty} z^r P\{N_t = r\} = b_t + \sum_{q=1}^{t} b_{t-q} \sum_{k=1}^{q} k! \, z^k \sum_{j=1}^{I_{k,q}} A_{k,q,j}, \qquad N > t \qquad (38)$$

To derive an expression for $P\{N_t = r\}$, $t = 1, 2, \ldots,$ $r = 0, 1, 2, \ldots,$ we will distinguish between two cases, one where $1 < N \leq t$ and the other where $N > t$

Case 1 $\underline{1 < N \leq t}$

From (37), we have

$$P\{N_t = 0\} = 0 \qquad (39)$$

$$P\{N_t = r\} = 0 \quad , \quad r = t + 1, t + 2, \ldots$$

$$P\{N_t = r\} = r! \sum_{q=t-N+1}^{t} b_{t-q} \sum_{j=1}^{I_{r,q}} A_{r,q,j} \quad , \quad r = 1, 2, \ldots, t$$

Case 2 $\underline{N > t}$

From (38), we have

$$P\{N_t = 0\} = b_t \qquad (40)$$

$$P\{N_t = r\} = 0 \quad , \quad r = t + 1, t + 2, \ldots$$

$$P\{N_t = r\} = r! \sum_{q=r}^{t} b_{t-q} \sum_{j=1}^{I_{r,q}} A_{r,q,j} \quad , \quad r = 1, 2, \ldots, t$$

It is clear from (40) that the distribution of the number of replacements in a time interval $(0, t]$ is the same for all values of N greater than t. In this case, while solving the Diophantine equations (20), (21) with r and q as parameters instead of υ and τ, we can conveniently set $N = t + 1$ in order to have a minimum number of unknowns, although any value of $N > t$ would yield the same final probability given by (40).

5.4 A Sample Calculation For $t = 10$, $N = 3$

Note: b_n as usual denotes $\phi^{(n)}(S)$, and the results are true for any distribution $\phi(\xi)$, $0 < \xi < \infty$ and any S

$$P\{N_t = r\} = 0 \quad , r = 0, 1, 2$$

$$P\{N_t = r\} = 0 \quad , r = 11, 12, \ldots$$

$$P\{N_t = 3\} = 3b_2^3(b_1 - b_2) + b_1 b_2^3$$

$$P\{N_t = 4\} = b_2[6(1-b_1)^2 b_2^2 + 12(1 - b_1)(b_1 - b_2)^2 b_2 + (b_1 - b_2)^4]$$

$$+ b_1[12(1 - b_1)(b_1 - b_2) b_2^2 + 4(b_1 - b_2)^3 b_2] +$$

$$[4(1 - b_1) b_2^3 + 6(b_1 - b_2)^2 b_2^2]$$

$$P\{N_t = 5\} = b_2[20(1 - b_1)^3 (b_1 - b_2) b_2 + 10(1 - b_1)^2 (b_1 - b_2)^3] +$$

$$+ b_1[10(1 - b_1)^3 b_2^2 + 30(1 - b_1)^2 (b_1 - b_2)^2 b_2 +$$

$$5(1 - b_1)(b_1 - b_2)^4] + [30(1 - b_1)^2 (b_1 - b_2) b_2^2 +$$

$$20(1 - b_1)(b_1 - b_2)^3 b_2 + (b_1 - b_2)^5]$$

$$P\{N_t = 6\} = b_2[6(1 - b_1)^5 b_2 + 15(1 - b_1)^4 (b_1 - b_2)^2] +$$

$$b_1[30(1 - b_1)^4 (b_1 - b_2)b_2 + 20(1 - b_1)^3 (b_1 - b_2)^3] +$$

$$[15(1 - b_1)^4 b_2^2 + 60(1 - b_1)^3 (b_1 - b_2)^2 b_2 +$$

$$15(1 - b_1)^2 (b_1 - b_2)^4]$$

$$P\{N_t = 7\} = b_2[7(1 - b_1)^6 (b_1 - b_2)] + b_1[7(1 - b_1)^6 b_2 +$$

$$21(1 - b_1)^5 (b_1 - b_2)^2] + [42(1- b_1)^5 (b_1 - b_2) b_2 +$$

$$35(1 - b_1)^4 (b_1 - b_2)^3]$$

$$P\{N_t = 8\} = b_2[1 - b_1]^8 + b_1[8(1 - b_1)^7 (b_1 - b_2)] +$$

$$[8(1 - b_1)^7 b_2 + 48(1 - b_1)^6 (b_1 - b_2)^2]$$

$$P\{N_t = 9\} = b_1[(1 - b_1)^9] + [9(1 - b_1)^8 (b_1 - b_2)]$$

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$$P\{N_t = 10\} = (1 - b_1)^{10}$$

5.5 The Mean Value of the Number of Replacements in a time Interval $(0, t]$.

The formal expressions obtained from (39) are

Case 1: $N \le t$

$$E[N_t] = \sum_{r=1}^{t} rP\{N_t = r\} = \sum_{r=1}^{t} r^2(r-1)! \sum_{q=t-N+1}^{t} b_{t-q} \sum_{j=1}^{I_{r,q}} A_{r,q,j} \quad (41)$$

Case 2: $N > t$

$$E[N_t] = \sum_{r=1}^{t} rP\{N_t = r\} = \sum_{r=1}^{t} r^2(r-1)! \sum_{q=r}^{t} b_{t-q} \sum_{j=1}^{I_{r,q}} A_{r,q,j} \quad (42)$$

6. An Economic Replacement Model

In this section we consider the problem of minimizing the total steady-state expected cost per period, which is the sum of the steady-state expected costs per period of replacement and operation. In formulating this cost objective function, S and N appear as the decision variables and optimal values of these will be obtained. It will be assumed that we have a fixed recurring cost of replacement per period and a linear expected operating cost per period. The analysis will be made for the case where the service aging per period, D_i ($i = 1, 2, \ldots$), has a gamma distribution. That for the negative exponential distribution will be derived as a special case.

To this end, we introduce the following notation:

$L(y, \theta)$ = conditional expected operating cost per period given that at the beginning of each period the equipment item has chronologial age θ, ($\theta = 1, 2, \ldots, N - 1$) and service age y ($0 < y \le S$)

$L(0, 0)$ = conditional expected operating cost per period given that the equipment item has just been replaced.

K = fixed cost of replacement

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R(S, N) = steady-state expected cost of replacement per period

O(S, N) = steady-state expected cost of operation per period

F(S, N) = steady-state total expected cost per period

$$= R(S, N) + O(S, N)$$

The results (10) and (11) obtained in Section 4 for the steady-state probability of a replacement being made, M, and for the limiting distribution of $\{(Y_t, \Theta_t); t = 1, 2, \ldots\}$ will now be used here in formulating the cost objective function, F(S, N).

We have, from (10)

$$R(S, N) = KM + ML(0, 0)$$

$$= \frac{K + L(0, 0)}{1 + \sum_{\theta=1}^{\theta=N-1} \phi^{(\theta)}(S)} \tag{43}$$

and from (11)

$$O(S, N) = \sum_{\theta=1}^{\theta=N-1} \int_0^S L(y, \theta)\, \Psi(y, \theta)\, dy \tag{44}$$

$$= \frac{\sum_{\theta=1}^{\theta=N-1} \int_0^S L(y, \theta)\, \phi^{*(\theta)}(y)\, dy}{1 + \sum_{\theta=1}^{\theta=N-1} \phi^{(\theta)}(S)}$$

where $\phi^{*(\theta)}(y)$ denotes as usual the θ-fold convolution of the density function $\phi(y)$ corresponding to the distribution function $\Phi(y)$.

From (43) and (44), we have,

$$F(S, N) = \frac{K + L(0, 0) + \sum_{\theta=1}^{\theta=N-1} \int_0^S L(y, \theta)\, \phi^{*(\theta)}(y)\, dy}{1 + \sum_{\theta=1}^{\theta=N-1} \int_0^S \phi^{*(0)}(y)\, dy} \tag{45}$$

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Assume a linear conditional expected operating cost in y and θ, and let

$$L(y, \theta) = A + By + C\theta \quad , 0 < y \leq S; \quad \theta = 1, 2, \ldots, N - 1 \tag{46}$$

$$L(0, 0) = A \tag{47}$$

Let the service aging per period have a gamma distribution with parameters r and λ i.e. let

$$\phi(y) = \begin{cases} 0 & -\infty < y \leq 0 \\ \\ \dfrac{(\lambda y)^{r-1} \lambda e^{-\lambda y}}{\Gamma(r)} & 0 < y < \infty \end{cases} \tag{48}$$

Then,

$$\phi^{*(\theta)}(y) = \begin{cases} 0 & -\infty < y \leq 0 \\ \\ \dfrac{(\lambda y)^{r\theta-1} \lambda e^{-\lambda y}}{\Gamma(r\theta)} & 0 < y < \infty \end{cases} \tag{49}$$

Defining $\gamma(a, x)$ to be the incomplete gamma function $\int_0^x e^{-t} t^{a-1} \, dt$, we have,

$$\int_0^S \phi^{*(\theta)}(y) dy = \frac{1}{\Gamma(r\theta)} \gamma(r\theta, \lambda S) \tag{50}$$

and, from (46), after some manipulations,

$$\int_0^S L(y, \theta) \phi^{*(\theta)}(y) dy = \int_0^S (A + By + C\theta) \phi^{*(\theta)}(y) dy \tag{51}$$

$$= \frac{1}{\Gamma(r\theta)} \left[(A + \frac{B}{\lambda} + C\theta) \gamma(r\theta, \lambda S) - \frac{B}{\lambda} \frac{(\lambda S)^{r\theta} e^{-\lambda S}}{r\theta} \right]$$

From (45), (47), (50) and (51), we have,

$$F(S, N) = \frac{K + A + \sum_{\theta=1}^{\theta=N-1} \frac{1}{\Gamma(r\theta)} \left[(A + \frac{B}{\lambda} + C\theta) \gamma(r\theta, \lambda S) - \frac{B}{\lambda} \frac{(\lambda S)^{r\theta} e^{-\lambda S}}{r\theta} \right]}{1 + \sum_{\theta=1}^{\theta=N-1} \frac{1}{\Gamma(r\theta)} \gamma(r\theta, \lambda S)} \tag{52}$$

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When the parameter r is an integer, then

$$\frac{\gamma(r\theta, \lambda S)}{\Gamma(r\theta)} = \sum_{i=r\theta}^{\infty} \frac{e^{-\lambda S}(\lambda S)^i}{i!} = P(r\theta; \lambda S) \tag{53}$$

where $P(r\theta; \lambda S)$ is defined to be the complementary cumulative Poisson with parameter λS.

In the special case when $r = 1$, we have a negative exponential distribution, and (52) becomes, after some algebra,

$$F(S, N) = \{K + A + \lambda S(A + C) + \frac{(\lambda S)^2}{2}(\frac{B}{\lambda} + C) - \tag{54}$$

$$[\lambda SA + \frac{(\lambda S)^2}{2}(\frac{B}{\lambda} + C)] P(N - 2; \lambda S) + [(N - 1) A - \lambda SC] P(N - 1; \lambda S)$$

$$+ [(\frac{B}{\lambda} + C) \frac{(N - 1)(N)}{2} P(N; \lambda S)\} / \{1 + \lambda S - \lambda S P(N - 2; \lambda S)$$

$$+ (N - 1) P(N - 1; \lambda S\} \qquad , N \geq 2$$

6.1 Computational Results

Owing to the complexity of expressions (52) and (54) for the total steady-state expected cost per period it is analytically difficult to obtain optimal values of N and S that minimize F(S, N), this difficulty being further compounded by the fact that one decision variable is discrete and the other continuous. In what follows several computational results are presented and discussed.

In Figures 1 and 2 the total steady state expected cost per period is plotted against discrete N for various fixed values of S (or, alternatively, λS). The resulting graphs have however been made continuous for convenience. The parameter r of the gamma distribution (48) is taken to be 1 and 3, respectively, so that Figure 1 actually represents the case when the service aging per period has a negative exponential distribution. In both graphs the values of the

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other input parameters are as follows: $\lambda = 10^{-3}$, K = 5000, A = 51, $B/_\lambda = 1$. and C = 300.

For the negative exponential distribution the optimal values of N and λS are found to be 6 and 17.0, respectively, whereas for the case, r = 3, (Fig. 2), these values are 6 and 35.0.

In general the patterns in Figures 1 and 2 exhibit a decreasing tendency, followed by an increasing one, this in turn giving way to a level stretch. For fixed λS, as the limit on the chronological age is raised, the equipment item tends to be replaced after a longer period of time and hence the expected cost of replacement decreases while the expected cost of operation increases with N. Once N is raised beyond a certain "high" value (this "high" being relative to the value of λS), however, it no longer dictates the replacement policy. It is the "lower" barrier of S that does so. Therefore, increasing N does not affect the expected costs of replacement and operation beyond this point. The behaviour of the graphs is now readily understandable from the fact that the total steady-state expected cost is nothing but the summation of the expected costs of replacement and operation.

For very low values of λS, N is irrelevant to the decision-making and the level stretch dominates early in the game. As λS is raised further, the increasing and decreasing portions of the graph both begin to make themselves felt. For very high values of λS, the curves converge, because now λS becomes irrelevant and N dominates the decision-making.

When r is increased from 1 in Fig. 1 to 3 in Fig. 2 the optimal values of N remains unchanged at 6 whereas that of λS increases from 17.0 to 35.0. This is explained by the fact that increasing r increases the expected value of the service age, given a certain chronological age θ at the beginning of a period.

In both cases when N is kept at its optimal value of 6, and λS is increased

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beyond its own optimal value of 17.0 or 35.0, the total steady-state expected cost per period remains unchanged because λS becomes "high" relative to this value of N and N dictates the replacement policy. Hence, it would be technically correct to say of Fig. 1, for example, that the optimum value of λS is greater than or equal to 17.0.

Both Figures 1 and 2 were plotted against N for various values of λS. It is easy to see, however, that the behaviour of the graphs would be essentially the same if the plots were instead made against λS for fixed values of N.

Varying the other input parameters would naturally affect the optimum values of N and S. If the value of C is lowered, for instance, to 30, (Fig. 3) the decreasing portion of the graph dominates, and the optimal values of N and λS are found to be as high as 18 and 30.0, respectively. It is obviously more economical to run the equipment for a longer period of time rather than incur a relatively exorbitant cost of replacement.
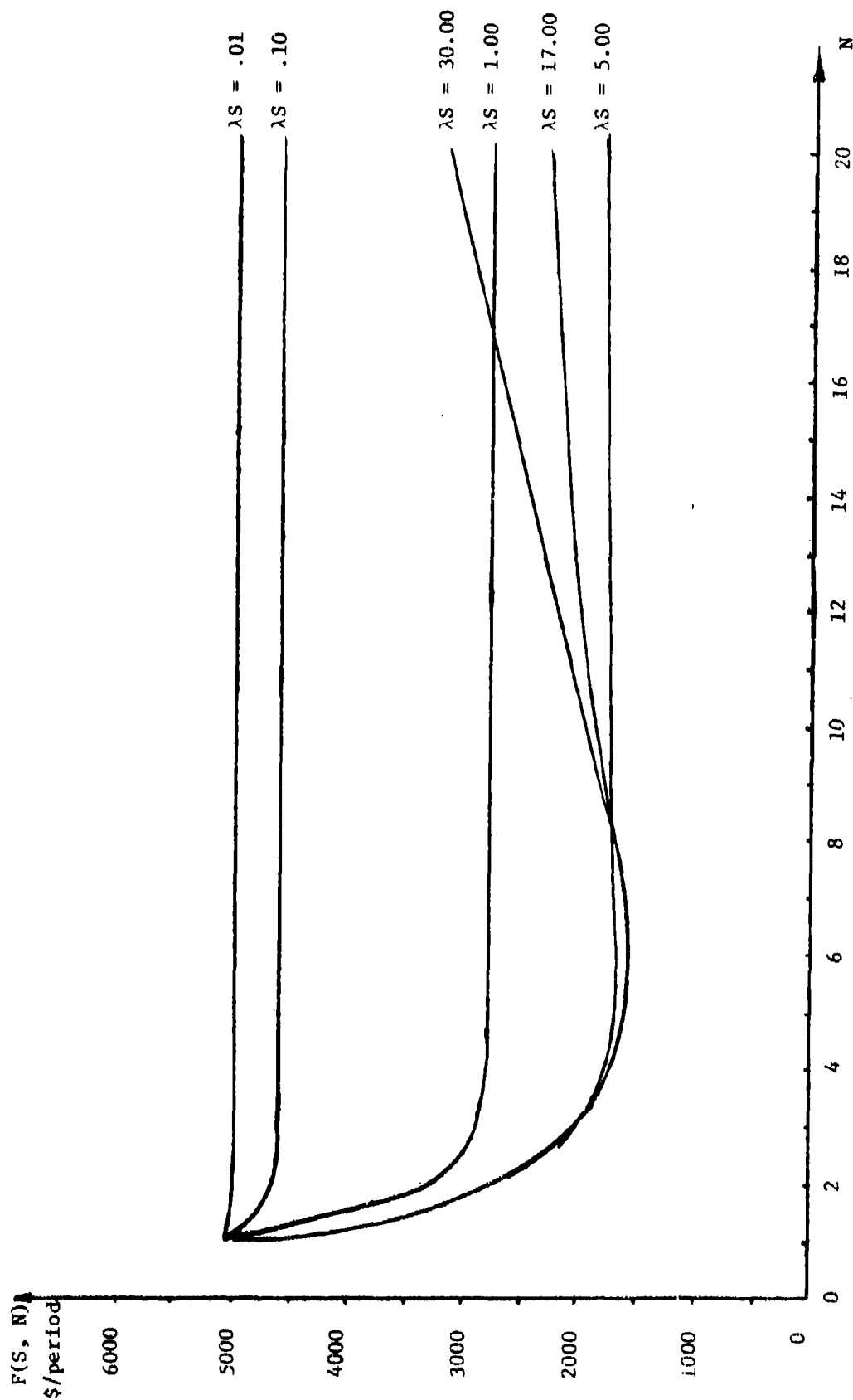
Figure 1: The function F(S, N) when demand has a negative exponential distribution ($\lambda = 10^{-3}$)

$K = 5,000$, $A = 51$, $B/\lambda = 1.0$, $C = 300$. $N^* = 6$, $\lambda S^* = 17.0$, $F(S^*, N^*) = 1636.83$
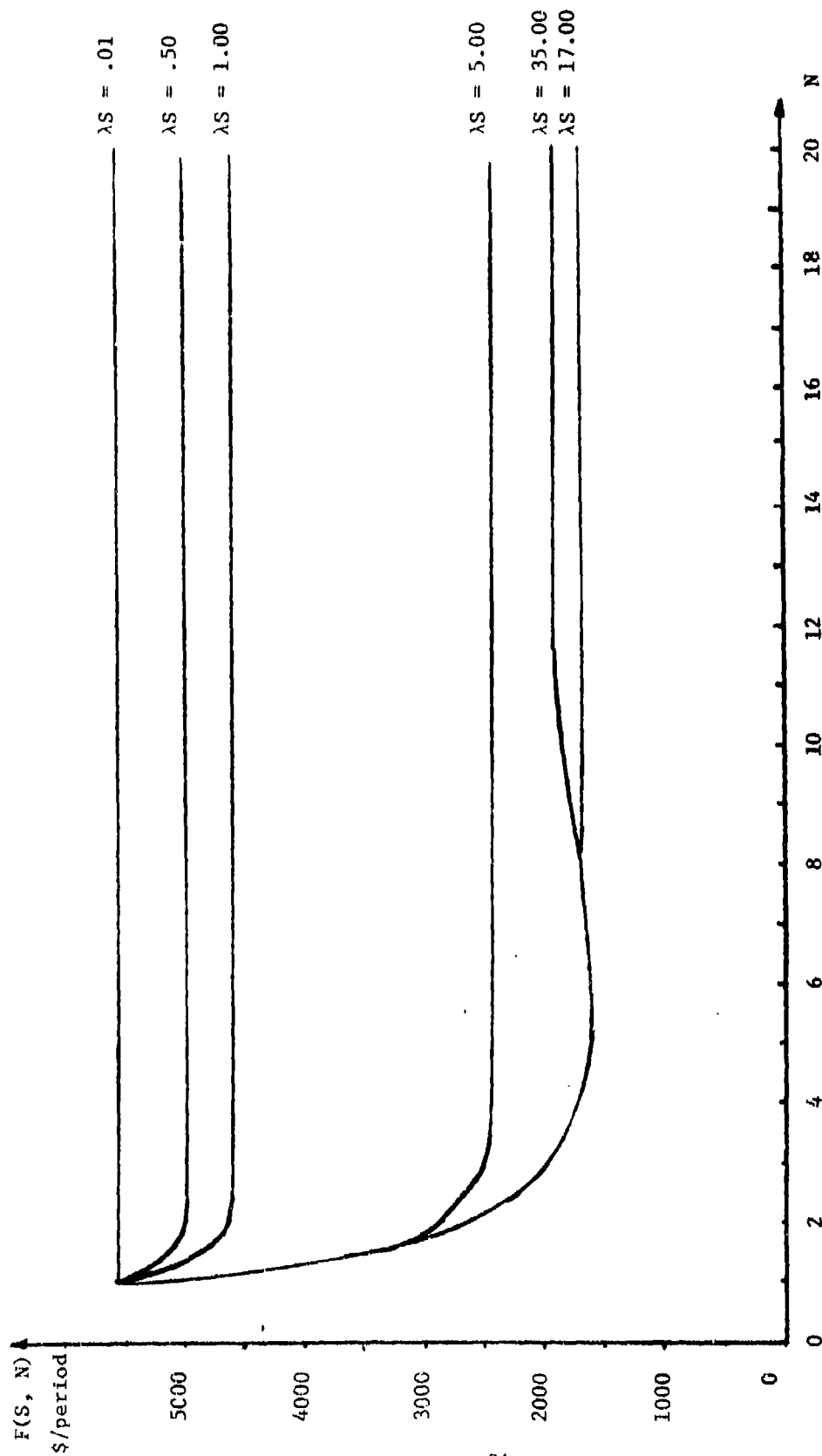
Figure 2: The function F(S, N) when demand has a gamma distribution (r = 3, $\lambda = 10^{-3}$)

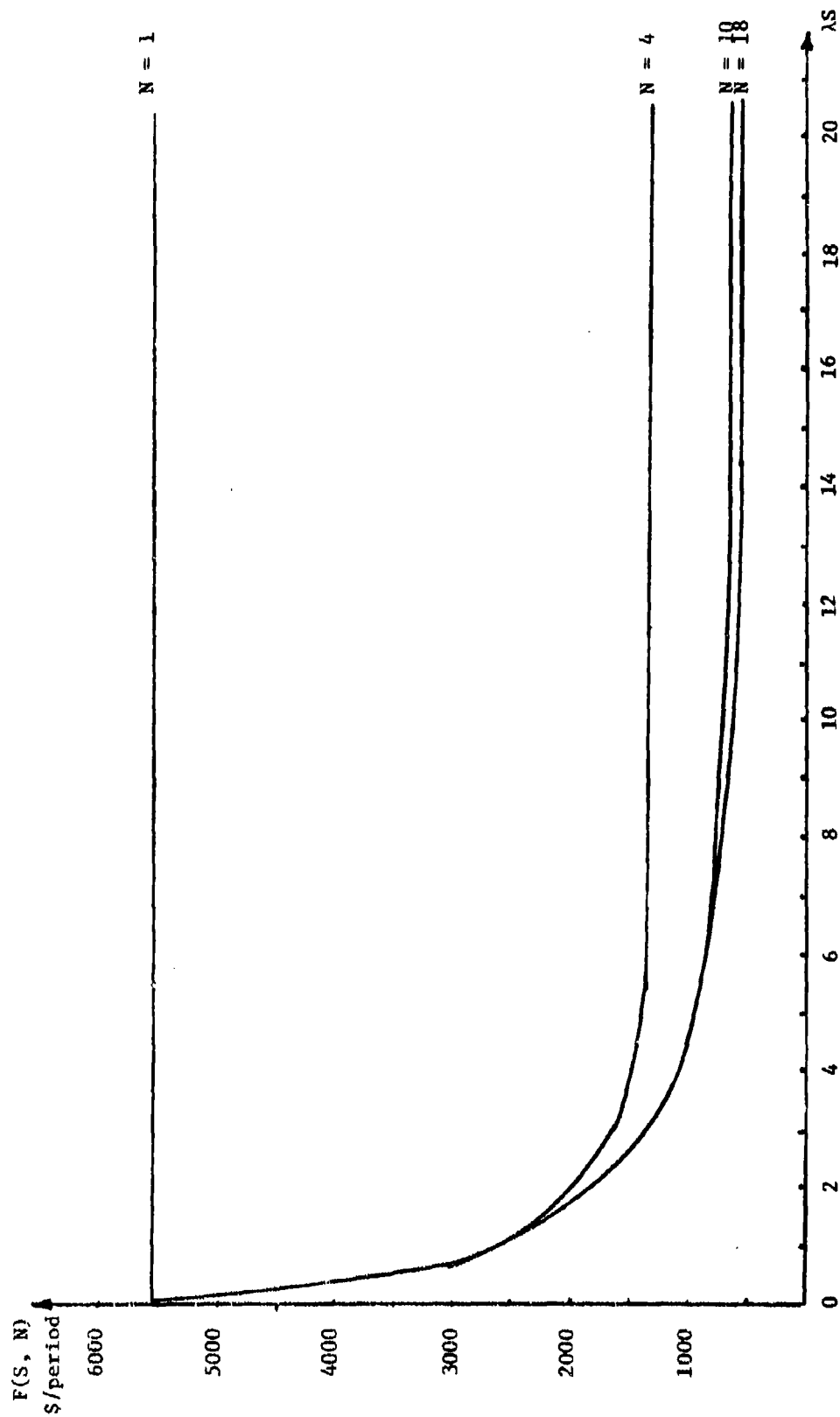K = 5,000, A = 51, B/λ = 1.0, C = 300. N^* = 6, λS^* = 35.0, $F(S^*, N^*)$ = 1635.16

Figure 3: The function F(S, N) when demand has a negative exponential distribution ($\lambda = 10^{-3}$)

$K = 5,000$, $A = 51$, $B/\lambda = 1.0$, $C = 30$. $N^* = 18$, $\lambda S^* = 30$, $F(S^*, N^*) = 592.27$

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